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Gluon condensate and c-quark mass in pseudoscalar sum rules at 3-loop order

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Abstract: Charmonium sum rules for pseudoscalar 0^{-+} state $\eta_c(1S)$ are analyzed within perturbative QCD and Operator Product Expansion. The perturbative part of the pseudoscalar correlator is considered at α_s^2 order and the contribution of the gluon condensate $\langle G^2 \rangle$ is taken into account with α_s correction. The OPE series includes the operators of dimension $D = 6, 8$ computed both in the instanton and factorization model. The method of moments in $\overline{\mathrm{MS}}$ scheme allows to establish acceptable values of the charm quark mass and gluon condensate, using the experimental mass of η_c. In result of the analisys the charm quark mass is found to be $\bar{m}_c = 1.26 \pm 0.02\,\mathrm{GeV}$ independently of the condensate value. The sensitivity of the results to various approximations for the massive 3-loop pseudoscalar correlator is discussed.

Keywords: QCD, Sum Rules.

Contents

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1. Introduction

The concept of Operator Product Expansion (OPE) was applied to QCD sum rules in [1] to parametrize the nonperturbative effects. The operators of increasing dimension, constructed from quark and gluon fields, or condensates, constitute the OPE series, which is added to the perturbative ones. In case of heavy quark correlators the quark condensates are not essential and OPE series start from the dimension $D = 4$ gluon condensate

$$\left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu} \right\rangle$$

for which the authors of [1] have obtained the estimation $0.012\,\mathrm{GeV}^4$ from vector charmonium sum rules. In [2] the charmonium sum rules were studied in pseudoscalar channel and it was predicted the mass of the lowest η_c state $3.00 \pm 0.03\,\mathrm{GeV}$. This result was in contradiction with available to that time experimental information. Later measurements found the mass of η_c close to $3.0\,\mathrm{GeV}$, which was considered as a triumph of QCD.

Since then various sum rules were analyzed in many publications[1] in order to obtain or specify the value of the gluon condensate. In the recent paper [3] the vector charmonium sum rules were reconsidered with α_s^2-corrections of the perturbative series and α_s-corrections to the condensate contribution, and up to date experimental data. The analysis of [3] resulted to the gluon condensate $0.009 \pm 0.007\,\mathrm{GeV}^4$ and c-quark mass $m_c(m_c) = 1.275 \pm 0.015\,\mathrm{GeV}$. Despite high accuracy of experimental data and c-quark mass determination, the accuracy of the gluon condensate remains $\sim \pm 100\%$, and zero value is not excluded.

[1] See [3] for the list of publications.

It also seems reasonable to reanalyze the pseudoscalar sum rule, taking into account the information obtained in [3]. Now the mass of η_c is known experimentally with high accuracy 2979.7 ± 1.5 MeV [4], so we invert the problem and find a restriction on the charm quark mass and gluon condensate, imposed by this sum rule. A special attention should be paid to the correlation between these two values, since a variation of one parameter leads to the change of another.

The sum rule technique goes as follows. The correlator of the pseudoscalar charm currents is defined as:

$$\Pi^p(q^2) \;=\; i \int dx\, e^{iqx} \,\langle\, T J^p(x) J^p(0)\,\rangle \qquad J^p = 2im\,\bar{c}\gamma_5 c\,. \tag{1.1}$$

We define the pseudoscalar current as $J^p = \partial_\mu J^a_\mu$, $J^a_\mu = \bar{c}\gamma_5\gamma_\mu c$ is axial vector current. Within the narrow width approximation the imaginary part is:

$$\mathrm{Im}\,\Pi^p(q^2 + i0) \;=\; \pi \sum_\eta \delta(q^2 - m_\eta^2)\; |\langle 0|J^p(0)|\eta\rangle|^2\;. \tag{1.2}$$

The sum goes over the pseudoscalar states[2] with $J^{PC} = 0^{-+}$. The correlator (1.1) is quadratically divergent, so the dispersion relation requires double subtraction:

$$\Pi^p(q^2) = c_0 + q^2 c_1 + \frac{q^4}{\pi}\int_0^\infty \frac{\mathrm{Im}\,\Pi^p(s+i0)}{s^2(s-q^2)}\,ds \tag{1.3}$$

provided the integral in the rhs is convergent, c_0, c_1 are unknown constants. In order to suppress the contribution of the higher states in (1.2) as well as continuum contribution, one considers the derivatives of the polarization operator in euclidean region $Q^2 \equiv -q^2 > 0$, the so-called moments:

$$\begin{aligned} M^p_n(Q^2) \equiv \frac{8\pi^2}{3n!}\left(-\frac{d}{dQ^2}\right)^n \Pi^p(Q^2) &= \frac{8\pi}{3}\int_0^\infty \frac{\mathrm{Im}\,\Pi^p(s+i0)}{(s+Q^2)^{n+1}}\,ds \\ &= \frac{8\pi^2}{3}\sum_\eta \frac{|\langle 0\,|J^p(0)|\,\eta\rangle|^2}{(m_\eta^2+Q^2)^{n+1}}\,, \end{aligned} \tag{1.4}$$

where $n \geq 2$. The matrix elements $\langle 0\,|J^p(0)|\,\eta\rangle$ are not known experimentally. But if one considers the ratio of some two moments at sufficiently high n, the contribution of the lightest state $\eta_c(1S)$ becomes dominant and

$$\frac{M^p_n(Q^2)}{M^p_{n+1}(Q^2)} = m^2_{\eta_c} + Q^2\,, \qquad n \to \infty\,. \tag{1.5}$$

This property was exploited to predict the mass of η_c in [2, 5]. An essential point was noticed in [5]: the QCD corrections to the moments are large at $Q^2 = 0$, so the sum rules should be considered at $Q^2 > 0$. Moreover, huge contribution of the dimension 8 operators $\langle G^4\rangle$ to the moments at $Q^2 = 0$ [7] becomes tolerable at $Q^2 \sim 4m_c^2$.

[2]The next to lightest pseudoscalar state $\eta_c(2S)$ with mass $3654 \pm 6(\mathrm{stat}) \pm 8(\mathrm{syst})$ MeV was discovered recently [6].

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The subject of this paper is a detailed analysis of the sum rule (1.5). In the next section 2 the perturbative and OPE corrections to the correlator (1.1) are described. Section 3 is devoted to the moments both in the pole and $\overline{\rm MS}$ scheme for the charm quark mass. In the section 4 various contribution to the pseudoscalar sum rule (1.5) are studied in details for typical values of the charm mass and gluon condensate. The higher dimension $D = 6, 8$ gluon operators are calculated both in the instanton and factorization model. In the final section 5 the restriction on the c-quark mass and the gluon condensate $\langle aG^2\rangle$ are obtained.

2. Pseudoscalar correlator in QCD

In QCD the polarization function (1.1) consists of perturbative part and operator product expansion:

$$\Pi^p = \Pi^p_{\rm PT} + \Pi^p_{\rm OPE}\,. \tag{2.1}$$

The perturbative part is determined by its imaginary part via dispersion relation (1.3). The imaginary part is parametrized by the coefficient functions $R^{(k),p}$ in the expansion by the running QCD coupling $a(\mu^2) \equiv \alpha_s(\mu^2)/\pi$:

$$\mathrm{Im}\,\Pi^p_{\rm PT}(s+i0) = \frac{3\,sm^2}{2\pi}\sum_{k\geq 0} R^{(k),p}(s,\mu^2)\,a^k(\mu^2)\,. \tag{2.2}$$

It is simpler to parametrize the functions $R^{(k),p}$ in terms of the pole masse m of c-quark. The first two terms do not depend on the scale μ^2. They are known analytically [8]:

$$\begin{aligned} R^{(0),p} &= v \\ R^{(1),p} &= \frac{v}{2}(7-v^2) + 4v\left(\ln\frac{1-v^2}{4} - \frac{4}{3}\ln v\right) + \frac{19+2v^2+3v^4}{12}\ln\frac{1+v}{1-v} + \frac{8}{3}(1+v^2)\times \\ &\times\left[2\,\mathrm{Li}_2\left(\frac{1-v}{1+v}\right) + \mathrm{Li}_2\left(-\frac{1-v}{1+v}\right) + \left(\frac{3}{2}\ln\frac{1+v}{2} - \ln v\right)\ln\frac{1+v}{1-v}\right]. \end{aligned} \tag{2.3}$$

Here and below $v = \sqrt{1-1/z}$, $z = s/(4m^2)$. The function $R^{(2),p}$ is usually decomposed into the following gauge invariant parts:

$$R^{(2),p} = C_F^2 R_A^{(2),p} + C_A C_F R_{NA}^{(2),p} + C_F T n_l R_l^{(2),p} + C_F T R_F^{(2),p} + C_F T R_S^{(2),p}\,, \tag{2.4}$$

where $C_A = 3$, $C_F = 4/3$, $T = 1/2$ are group constants and $n_l = 3$ is the number of light quarks. The function $R_l^{(2),p}$ comes from the diagram with massless quark loop. It was found in [8] and in our normalization takes the form:

$$R_l^{(2),p} = \left(-\frac{1}{4}\ln\frac{\mu^2}{4s} - \frac{5}{12}\right)R^{(1),p} + \delta_P^{(2)}\,, \tag{2.5}$$

where the function $\delta_P^{(2)}$ is given by equation (110) in ref. [8]. The function $R_F^{(2),p}$ comes from the diagram with 2 massive quark loops. For $s < 16m^2$ it contains only the contribution of virtual massive quarks and has the form [8]:

$$R_F^{(2),p} = 2v\,\mathrm{Re}\,P_Q^{(2)} - \frac{1}{4}R^{(1),p}\ln\frac{\mu^2}{m^2}\,, \qquad s < 16m^2\,, \tag{2.6}$$

where $P_Q^{(2)}$ is second order correction to the pseudoscalar current vertex from the diagram with massive quark loop; it is given by [8, equation (169)]. For $s > 16m^2$ the 4-particle cut must be included in $R_F^{(2),p}$. It is given by the double integral, in [8, eq. (97)], which cannot be taken analytically. Here, however, the total function $R_F^{(2),p}$ can be replaced by its high-energy asymptotic, available to the terms m^8/s^4 in [9].

The functions $R_A^{(2),p}$ and $R_{NA}^{(2),p}$ correspond to the diagrams with single massive quark loop and various gluon exchanges. They are not known analytically, so one has to use some approximations. It turns out, that the moments, computed by the dispersion relation (1.4), are sensitive to the choice of these approximations. The accuracy of the moments becomes especially important in $\overline{\text{MS}}$ scheme, where there is a sufficient cancellation between large terms (see eq. (3.9) below), so we describe this point in details.

The first 8 moments M_2–M_9 at $Q^2 = 0$ are known analytically [10]. We will require, that the approximations for $R_A^{(2),p}$ and $R_{NA}^{(2),p}$ *must reproduce these moments* with high accuracy being substituted into the dispersion integral in (1.4).[3] As usual, we shall apply the conformal mapping and Pade approximation for the relevant parts of the polarization function Π^p and take the imaginary part after then, see appendix A for details. Although such approximations are constructed so that they reproduce low-q^2 expansion of the polarization function, they do not give exact values of the first 8 moments at $Q^2 = 0$, computed by taking the dispersion integral in (1.4). Indeed, the Pade approximations have extra poles away from the cut $z = [1, \infty)$ and, strictly speaking, the dispersion relation (1.3) is not valid for them. The approximated formulas for $R_A^{(2),p}$ and $R_{NA}^{(2),p}$, used in this paper, are given in the equation (A.10) of appendix A.

The last term in (2.4) $R_S^{(2),p}$ is the so-called singlet part with 2 triangle quark loops. This part contains the 2-gluon cut, which is proportional to the 2-photon decay width of the pseudoscalar boson. It is known analytically [12, 13]:

$$R_{gg}^{(2),p} = TC_F \frac{|f(z)|^2}{2z}\,, \tag{2.7}$$

where

$$f(z) = -\frac{1}{2}\int_0^1 \frac{dx}{x}\ln\left[1 - 4zx(1-x)\right] = \begin{cases} \arcsin^2\sqrt{z}\,, & z < 1 \\ -\dfrac{1}{4}\left(\ln\dfrac{1+v}{1-v} - i\pi\right)^2, & z > 1\,. \end{cases}$$

The contribution of purely gluonic states to the heavy quark current correlators was discussed in details in [14] (3-gluon state in case of vector currents). In the narrow width approximation (1.2) only charmed states are taking into account. Since the 2-gluon state is not associated with any charmonium state, we subtract $R_{gg}^{(2),p}$ in the dispersion relation (1.3) and take the integral from $s = 4m^2$, in accordance with suggestion of [14]. The approximation for $R_S^{(2),p}$ without the 2-gluon cut is given in the eq. (A.10) of appendix A.

The leading in α_s order operator series Π^p_{OPE} (2.1) for the heavy quark correlator has been computed in [15] up to operators of dimension $D = 8$. This series can be compactly

[3]We found the approximations proposed in [10, eqs. (39), (40)], having insufficient accuracy to satisfy these requirement.

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expressed in terms of Gauss hypergeometric functions:

$$\Pi^p_{\rm OPE}(s) = \frac{1}{2\pi^2}\sum_{k\geq 2}\sum_j \frac{O_k^{(j)}}{(4m^2)^{k-2}}\sum_{i=1}^{2k-1}\frac{c^{(p,j)}_{k,i}}{k!\,(1/2)_{k-1}}\,{}_2F_1\left(\begin{matrix}1,\, k+i-2\\ k-1/2\end{matrix}\,\middle|\,\frac{s}{4m^2}\right), \tag{2.8}$$

where $(1/2)_{k-1}$ is Pochhammer symbol, $O_k^{(j)}$ are the operators of dimension $D = 2k$. For the heavy quark correlators there is single operator of dimension $D = 4$

$$O_2 = \langle g^2 G^a_{\mu\nu} G^a_{\mu\nu}\rangle \tag{2.9}$$

and 2 operators of dimension $D = 6$:

$$O_3^{(1)} = \left\langle g^3 f^{abc} G^a_{\mu\nu} G^b_{\nu\lambda} G^c_{\lambda\mu}\right\rangle\,, \qquad O_3^{(2)} = \langle g^4 j^a_\mu j^a_\mu\rangle\,, \tag{2.10}$$

where $g j^a_\mu = G^a_{\mu\nu;\,\nu} = \frac{g}{2}\sum_{q=u,d,s}\bar q\gamma_\mu\lambda^a q$. We choose 7 independent operators of dimension $D = 8$ according to [15]:

$$\begin{aligned}
O_4^{(1)} &= \left\langle \left(g^2 d^{abc} G^b_{\mu\nu} G^c_{\alpha\beta}\right)^2 + \frac{2}{3}\left(g^2 G^a_{\mu\nu} G^a_{\alpha\beta}\right)^2\right\rangle, \qquad O_4^{(2)} = \left\langle \left(g^2 f^{abc} G^b_{\mu\nu} G^c_{\alpha\beta}\right)^2\right\rangle,\\
O_4^{(3)} &= \left\langle \left(g^2 d^{abc} G^b_{\mu\alpha} G^c_{\alpha\nu}\right)^2 + \frac{2}{3}\left(g^2 G^a_{\mu\alpha} G^a_{\alpha\nu}\right)^2\right\rangle, \qquad O_4^{(4)} = \left\langle \left(g^2 f^{abc} G^b_{\mu\alpha} G^c_{\alpha\nu}\right)^2\right\rangle,\\
O_4^{(5)} &= \left\langle g^5 f^{abc} G^a_{\mu\nu} j^b_\mu j^c_\nu\right\rangle, \quad O_4^{(6)} = \left\langle g^3 f^{abc} G^a_{\mu\nu} G^b_{\nu\lambda} G^c_{\lambda\mu;\,\alpha\alpha}\right\rangle, \quad O_4^{(7)} = \langle g^4 j^a_\mu j^a_{\mu;\,\alpha\alpha}\rangle
\end{aligned} \tag{2.11}$$

The coefficients $c^{(p,j)}_{k,i}$ in (2.8) can be obtained from [15]:

$$c^{(p)}_{2,i} = (0,\, 1/4,\, -1/12)\,, \qquad c^{(p,j)}_{3,i} = $$

$j\backslash i$	1	2	3	4	5
1	0	$-3/2$	5	$-22/5$	1
2	1/3	$-8/3$	3	28/15	$-4/3$

$$c^{(p,j)}_{4,i} = \tag{2.12}$$

$j\backslash i$	1	2	3	4	5	6	7
1	1/3	31	-258	1580/3	-365	70	0
2	1/6	10	-111	946/3	$-4873/14$	144	-10
3	$-4/3$	-16	168	$-596/3$	-106	140	0
4	1/3	122	-946	5480/3	$-7435/7$	-20	92
5	$-10/3$	12	260	$-2888/3$	7290/7	-304	-24
6	0	-22	152	-174	$-1108/7$	296	-96
7	2	-36	100	-128	1314/7	-208	72

The α_s correction to the $D = 4$ condensate contribution was obtained analytically in [16]. One could differentiate it n times to obtain the moments. However, we prefer to use a dispersion-like relation for this correction, constructed in appendix B, which is convenient for numerical calculation of the moments, especially for high n.

3. Moments in $\overline{\mathrm{MS}}$ scheme

At first let us consider the moments in the pole-mass scheme. In QCD the moments (1.4) are expanded by the running QCD coupling $\alpha_s(\mu^2)$. The approximation, used in this paper, includes the following ingredients: 1) the perturbative series up to α_s^2 order, 2) the operator series up to dimension $D = 8$, and 3) α_s correction to the $\langle G^2 \rangle$ operator contribution. Adding all pieces together, we write down the following expression for the pseudoscalar moments:

$$M_n^p(Q^2) = \sum_{k=0}^{2} M_n^{(k),p}(Q^2)\, a^k(m^2) + \sum_{k=2}^{4}\sum_{j} O_k^{(j)}\, M_n^{(Ok,j),p}(Q^2) + O_2\, M_n^{(O2)(1),p}(Q^2)\, a(m^2)\,, \tag{3.1}$$

for definiteness the coupling $a \equiv \alpha_s/\pi$ is taken at the scale $\mu^2 = m^2$. As discussed in previous section, the perturbative moments are taken without 2-gluon cut (2.7):

$$M_n^{(k),p}(Q^2) = 4m^2 \int_{4m^2}^{\infty} \frac{s\,(R^{(k),p} - R_{gg}^{(k),p})(s,m^2)}{(s+Q^2)^{n+1}}\, ds\,. \tag{3.2}$$

The leading order can be expressed in terms of Gauss hypergeometric function:

$$M_n^{(0),p}(Q^2) = \frac{1}{(4m^2)^{n-2}}\,\frac{\Gamma(n-1)}{2\,(1/2)_n}\; {}_2F_1\left(\begin{matrix} n-1,\, n+1 \\ n+1/2 \end{matrix}\,\middle|\, -\frac{Q^2}{4m^2}\right), \tag{3.3}$$

where $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ is Pochhammer symbol. The higher order perturbative moments are computed numerically by (3.2).

The contribution of the operators $O_k^{(j)}$ to the moments can be easily obtained by differentiating eq (2.8):

$$M_n^{(Ok,j),p}(Q^2) = \frac{4}{3}\sum_{i=1}^{2k-1} \frac{c_{k,i}^{(p,j)}}{(4m^2)^{n+k-2}}\,\frac{(k+i-2)_n}{k!\,(1/2)_{n+k-1}}\; {}_2F_1\left(\begin{matrix} n+1,\, n+k+i-2 \\ n+k-1/2 \end{matrix}\,\middle|\, -\frac{Q^2}{4m^2}\right). \tag{3.4}$$

The α_s-correction to the $D = 4$ gluon condensate contribution can be obtained by differentiating eq (B.7) of appendix B:

$$M_n^{(O2)(1),p}(Q^2) = \frac{2}{3(4m^2)^n}\sum_{i=1}^{3} \frac{(n+1)_{i-1}}{(i-1)!}\left[\frac{\pi^2 f_i^p}{(1+y)^{n+i}} + \int_1^{\infty} \frac{F_i^p(z)}{(z+y)^{n+i}}\, dz\right], \tag{3.5}$$

where $y = Q^2/(4m^2)$, the constants f_i^p and the functions $F_i^p(z)$ are given in eqs (B.4) and (B.6) of appendix B. Notice, that eqs (3.2)–(3.5) are applicable for noninteger n also.

Similarly to the vector case [3], the α_s-corrections to the moments are unacceptably large in the pole mass scheme and the series (3.1) is divergent. The pole mass, in fact, is the mass of free quark. Since the quarks exist only in form of strongly bounded states, the physical meaning of the pole quark mass is rather unclear; it cannot be found from the sum rules with a good accuracy.

Instead of the pole mass one introduces another effective mass parameter, to improve the convergence of the perturbative series. Authors of [2, 5] used the mass, renormalized

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at the euclidean point $p^2 = -m^2$. In this paper we shall use the most popular choice for today: the gauge invariant mass in the modified minimal subtraction ($\overline{\rm MS}$) scheme taken at the scale, equal to the mass itself $\bar{m} \equiv \bar{m}(\bar{m}^2)$. The pole mass m is perturbatively expressed in terms of $\bar{m}$:

$$\frac{m^2}{\bar{m}^2} = 1 + \sum_{n\geq 1} K_n\, a^n(\bar{m}^2)\,. \tag{3.6}$$

The 2-loop factor was found, in particular, in [17] while the 3-loop factor was recently calculated in [18]:

$$\begin{aligned} K_1 &= \frac{8}{3} \\ K_2 &= 28.6646 - 2.0828\, n_l = 22.4162 \\ K_3 &= 417.039 - 56.0871\, n_l + 1.3054\, n_l^2 = 260.526\,. \end{aligned} \tag{3.7}$$

We put $n_l = 3$ in the last column. Now we reexpand the moments (3.1) by the QCD coupling $a(\bar{m}^2)$:

$$M_n^p(Q^2) = \sum_{k=0}^{2} \bar{M}_n^{(k),p}(Q^2)\, a^k(\bar{m}^2) + \sum_{k=2}^{4}\sum_j O_k^{(j)}\, \bar{M}_n^{(Ok,j),p}(Q^2) + O_2\, \bar{M}_n^{(O2)(1),p}(Q^2)\, a(\bar{m}^2)\,, \tag{3.8}$$

where $\bar{M}_n^{(k),p}(Q^2)$ are expressed in terms of $M_n^{(k),p}(Q^2)$ [3]:

$$\begin{aligned} \bar{M}_n^{(0),p}(Q^2) &= M_n^{(0),p} \\ \bar{M}_n^{(1),p}(Q^2) &= M_n^{(1),p} - K_1(n-d/2)\, M_n^{(0),p} + K_1(n+1)\, Q^2 M_{n+1}^{(0),p} \\ \bar{M}_n^{(2),p}(Q^2) &= M_n^{(2),p} - K_1\left(n-\frac{d}{2}\right) M_n^{(1),p} + K_1(n+1)\, Q^2 M_{n+1}^{(1),p} + \\ &\quad + \left(n-\frac{d}{2}\right)\left[\frac{K_1^2}{2}\left(n+1-\frac{d}{2}\right) - K_2\right] M_n^{(0),p} + \\ &\quad + (n+1)\left[K_2 - K_1^2\left(n+1-\frac{d}{2}\right)\right] Q^2 M_{n+1}^{(0),p} + \\ &\quad + \frac{K_1^2}{2}(n+1)(n+2)\, Q^4 M_{n+2}^{(0),p} \\ \bar{M}_n^{(Ok,j),p}(Q^2) &= M_n^{(Ok,j),p} \\ \bar{M}_n^{(O2)(1),p}(Q^2) &= M_n^{(O2),p} - K_1\left(n+2-\frac{d}{2}\right) M_n^{(G,0),p} + K_1(n+1)\, Q^2 M_{n+1}^{(G,0),p}\,, \end{aligned} \tag{3.9}$$

where $d = 4$ is the dimension of the pseudoscalar function $\Pi^p(Q^2)$, all $M_n^{(i),p}$ in the rhs are computed with $\overline{\rm MS}$ mass $\bar{m}$. The series (3.8) is much better convergent than (3.1). The numerical values of the ratios $\bar{M}^{(1,2)}/\bar{M}^{(0)}$ and $\bar{M}^{(O2)(1)}/\bar{M}^{(O2)}$ for $Q^2/(4\bar{m}^2) = 0, 1, 2$ and $n = 2 - 30$ are given in the table 1 of appendix C. Notice, that the values of $\bar{M}^{(2)}$ are approximate; other approximations for $R^{(2)}$ may lead to the moments $\bar{M}^{(2)}$, which differ from the numbers of the table 1 within $5 - 10\%$.

The expansion (3.8) goes by $a(\bar{m}^2)$. If one takes the QCD coupling at some another scale μ^2, the function $\bar{M}^{(2),p}$ changes:

$$a(\bar{m}^2) \to a(\mu^2)\,, \qquad \bar{M}_n^{(2),p}(Q^2) \to \bar{M}_n^{(2),p}(Q^2) + \bar{M}_n^{(1),p}(Q^2)\,\beta_0 \ln\frac{\mu^2}{\bar{m}^2} \tag{3.10}$$

so that the series (3.8) is μ^2-independent at the order α_s^2.

4. Pseudoscalar sum rule

It is convenient to define a dimensionless ratio of the pseudoscalar moments:

$$r_n(Q^2) = \frac{1}{4\bar{m}^2}\frac{M_n^p(Q^2)}{M_{n+1}^p(Q^2)} \to \frac{m_{\eta_c}^2 + Q^2}{4\bar{m}^2}\,, \qquad n \to \infty\,. \tag{4.1}$$

Theoretical ratio depends on the quark mass $\bar{m}$, QCD coupling α_s and condensates. But if the dimensionless parameters $Q^2/(4\bar{m}^2)$, $\langle aG^2\rangle/(4\bar{m}^2)^2$ etc. are fixed, the l.h.s. of (4.1) does not depend on the quark mass $\bar{m}$ (in fact, the QCD coupling depends on the scale, which itself may depend on $\bar{m}$; but this dependence is weak within the range of error of $\bar{m}$). So one may use the ratio (4.1) to find the $\overline{\rm MS}$ charm quark mass $\bar{m}$ for given condensates and QCD coupling.

The QCD coupling constant α_s is universal value and can be taken from other experiments. As input parameter, it is convenient to take α_s at the τ-lepton mass [4]:

$$\alpha_s(m_\tau^2) = 0.33 \pm 0.03\,, \qquad m_\tau = 1.777\,\text{GeV}\,. \tag{4.2}$$

Using this value as the boundary condition in the renormalization group equation, the QCD coupling can be evaluated at any scale. As argued in [3], the most natural scale for α_s is

$$\mu^2 = Q^2 + \bar{m}^2\,. \tag{4.3}$$

Indeed, in the limit $Q^2 \gg \bar{m}^2$ we come to natural massless choice $\alpha_s(Q^2)$, while at $Q^2 = 0$ it becomes $\alpha_s(\bar{m}^2)$. Later we shall vary the scale (4.3) to check the stability of results.

The $\overline{\rm MS}$ charm quark mass is determined from vector charmonium sum rules with high accuracy. The analysis of the moments at $Q^2 = 0$ with α_s^2 corrections leads to $\bar{m} = 1.304 \pm 0.027\,\text{GeV}$ in [19] and $1.23 \pm 0.09\,\text{GeV}$ in [20]. The authors of [19] neglected the condensate contribution, while in [20] the value $\langle aG^2\rangle = 0.024 \pm 0.012\,\text{GeV}^4$ was employed. In fact, the gluon condensate weakly affects on the mass value. But for the condensate determination the mass accuracy is especially important: a small mass variation leads to significant condensate change. As noticed in [3], the perturbative α_s and α_s^2 corrections to the vector moments $M_n(Q^2)$ in $\overline{\rm MS}$ scheme are strongly suppressed for $Q^2/(4\bar{m}^2) \approx n/5 - 1$ and $n > 5$. The analysis of [3] at $Q^2/(4\bar{m}^2) = 1, 2$ allowed to determine the c-quark mass with high acuracy:

$$\bar{m}(\bar{m}) = 1.275 \pm 0.015\,\text{GeV} \tag{4.4}$$

independently of the condensate value. (If the condensate is fixed, the error in (4.4) can be reduced even further.) This result is close to the recent lattice calculation $\bar{m} = 1.26 \pm 0.04(\text{stat}) \pm 0.12(\text{syst})\,\text{GeV}$ [21].

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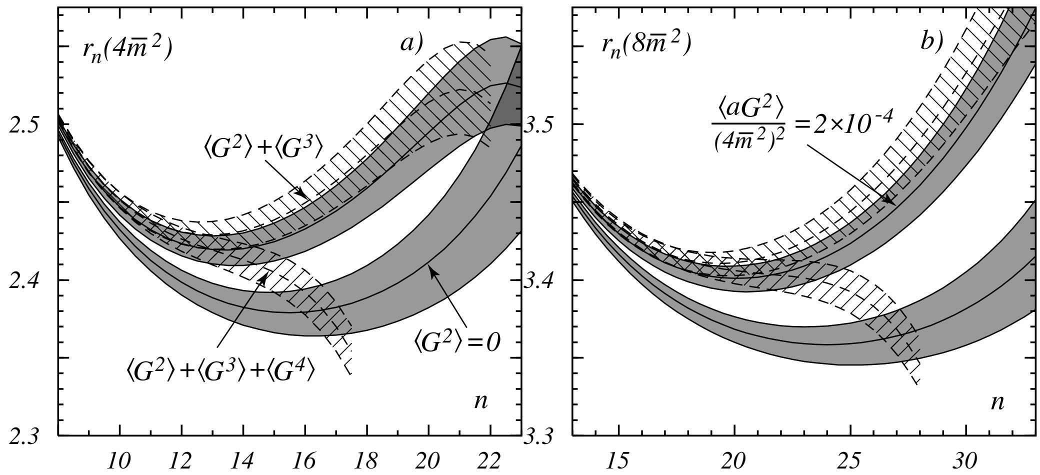

Figure 1: Ratio $r_n(Q^2)$ for $Q^2 = 4\bar{m}^2$ (a) and $Q^2 = 8\bar{m}^2$ (b) versus n. Lower shaded curve is purely perturbative, upper shaded curve is computed with condensate $\langle aG^2\rangle/(4\bar{m}^2)^2 = 2\times10^{-4}$. Hatched curves include the $\langle G^3\rangle$ operator (upper curve) and $\langle G^3\rangle + \langle G^4\rangle$ operators (lower curve) computed in the instanton model (4.6). The errorband of each curve corresponds to the error of α_s in (4.2).

For the mass (4.4) one gets the ratio (4.1)

$$r_n(Q^2) = \frac{Q^2}{4\bar{m}^2} + 1.37 \pm 0.03 \tag{4.5}$$

in the limit $n \to \infty$. Which values of $Q^2/(4\bar{m}^2)$ are convenient for the sum rule analysis? The choice $Q^2 = 0$ is not appropriate, since the perturbative corrections to the moments are large for almost all n, even in $\overline{\text{MS}}$ scheme. Large $Q^2 \gtrsim 12\bar{m}^2$ are also dangerous: in particular, when one changes the scale of α_s in (3.10), the effective expansion parameter $a\beta_0 \ln(Q^2/\bar{m}^2)$ becomes large $\gtrsim 0.5$. In what follows we shall use two choices $Q^2/(4\bar{m}^2) = 1, 2$.

The theoretical ratios $r_n(4\bar{m}^2)$ and $r_n(8\bar{m}^2)$ are plotted versus n in the figure 1(a) and 1(b) respectively. The lower shaded curve is purely perturbative, i.e. for $\langle aG^2\rangle = 0$. The central line of the shaded area corresponds to the central value of α_s (4.2), the errorband covers the error of the coupling α_s in (4.2). One sees, that the agreement with (4.5) is achieved within relatively narrow range of n: $n \sim 16$ for $Q^2 = 4\bar{m}^2$ and $n \sim 24$ for $Q^2 = 8\bar{m}^2$. If we look at the table 1, the perturbative corrections to the moments in $\overline{\text{MS}}$ scheme, as well as α_s correction to the condensate contribution, are minimal here. For higher n these corrections grow rapidly and the perturbation theory cannot be trusted here. For lower n the perturbative corrections are also large, and the leading order of the $D = 4$ condensate contribution crosses 0 at some point, so the behavior of the α_s-series is rather unclear here. Moreover, unknown contribution of $\eta_c(2S)$ and higher states to the experimental moments could be significant for low n.

Now we consider nonzero $D = 4$ condensate. As an illustration, let us fix the ratio $\langle aG^2\rangle/(4\bar{m}^2)^2 = 2\times10^{-4}$, which corresponds to $\langle aG^2\rangle \approx 0.008\,\text{GeV}^4$, close to the central value obtained in [3]. The ratio r_n with this condensate is shown by the upper shaded curves in figure 1. The ratio becomes higher for nonzero condensate, which tells in favor of lower mass of c-quark. At $Q^2 = 4\bar{m}^2$ the ratio is even higher, than (4.5) for all n.

Several models were employed to estimate the higher dimension $D = 6, 8$ operators. Here we consider the dilute instanton gas model [22] and vacuum dominance (or factorization) model.

Instanton model. The vacuum configuration is codsidered as a dilute gas of noninteracting instantons with effective radis ρ_c and concentration n_0. The radius ρ_c varies within $0.3 - 1\,\mathrm{fm}$ in the literature. Here we shall use the estimation $\rho_c = 0.5\,\mathrm{fm}$ obtained in [23]. The instanton concentration n_0 is fitted to the $D = 4$ gluon condensate $\left\langle aG^2 \right\rangle = 32\, n_0$. Then one obtains the following expressions for the $D = 6, 8$ gluon condensates (2.10) and (2.11):

$$O_3^{(1)} = \frac{12}{5\rho_c^2}\, O_2\,, \qquad O_3^{(2)} = 0\,,$$
$$\left(O_4^{(1)}, \ldots, O_4^{(7)}\right) = (4, 8, 3, 4, 0, 8, 0)\, \frac{16}{7\rho_c^4}\, O_2\,. \tag{4.6}$$

The ratio r_n with the operators $O_{3,4}$ computed by the instanton model (4.6) is show by hatched curves in figure 1. Upper hatched curve includes $\left\langle G^3 \right\rangle$ operator only, lower hatched curve includes both $\left\langle G^3 \right\rangle$ and $\left\langle G^4 \right\rangle$.

The $\left\langle G^3 \right\rangle$ operator contribution is small. But the contribution of the $\left\langle G^4 \right\rangle$ operators is large in the region of interest. Obviously the place, where the lower hatched curve crosses the perturbative one (lower shaded), the sum rule (4.1) with the operators (4.6) is not applicable. Here the $\left\langle G^4 \right\rangle$ contribution exceeds the leading order $\left\langle G^2 \right\rangle$, and the OPE series diverges.

It is a demonstration, that the higher order operators are essentialy overestimated in the instanton model [23]. Moreover, their values strongly depend on the instanton size ρ_c, which is not strictly fixed. For this reason we finish the analysis in the instanton model. The main outcome of this analysis is relatively small contribution of the operator $\left\langle G^3 \right\rangle$, which will be ignored in what follows.

Factorization hypothesis . In the factorization model the $\langle G^4 \rangle$ operators are proportional to $(\langle G^2 \rangle)^2$. The operators with the light quark current j^a_μ in (2.10), (2.11) can also be estimated by the factorization, but their size is much smaller. The operator $O_4^{(6)}$ with derivatives was taken as $O_4^{(6)} \approx M^2 O_3^{(1)}$ in [15], where $M^2 \approx 0.3\,\mathrm{GeV}^2$ characterizes the gluon virtuality in the vacuum. Alternatively, one may express this operator as

$$O_4^{(6)} = 2\left\langle g^4 f^{abc} G^a_{\mu\nu} G^b_{\nu\lambda} j^c_{\lambda;\mu} \right\rangle + 2\, O_4^{(4)}\,.$$

Since we neglect the operators with j^a_μ, we take $O_4^{(6)} = 2 O_4^{(4)}$ here (both estimations agree in the order of magnitude for typical condensates). Summarizing, we write down the $D = 8$ operators as:

$$\left(O_4^{(1)}, \ldots, O_4^{(7)}\right) = \left(\frac{65}{144}, \frac{5}{16}, \frac{19}{72}, \frac{1}{16}, 0, \frac{1}{8}, 0\right) (O_2)^2\,. \tag{4.7}$$

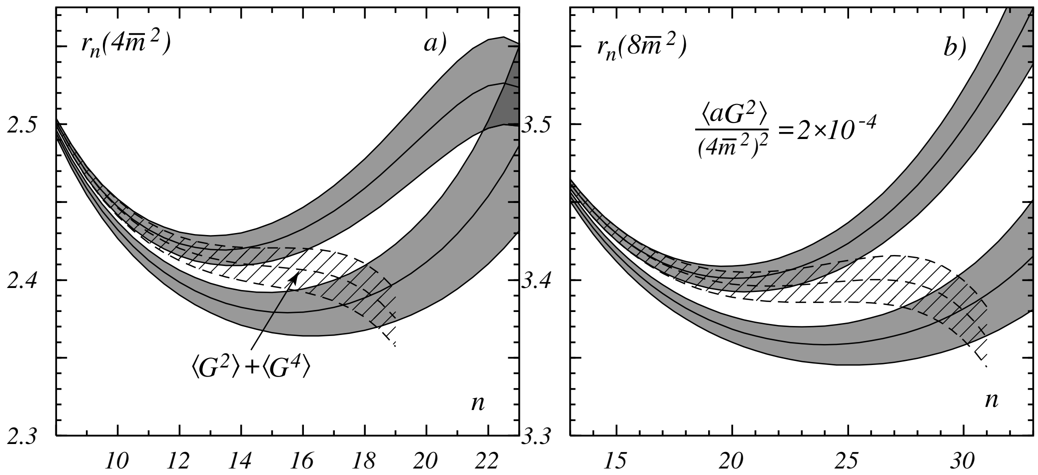

Figure 2: Ratio $r_n(Q^2)$ for $Q^2 = 4\bar{m}^2$ (a) and $Q^2 = 8\bar{m}^2$ (b) versus n in the factorization model. The $\langle G^3\rangle$ condensate is neglected, the contribution of the $\langle G^4\rangle$ operators according to (4.7) is displayed by the hatched curve. Other notations are the same as in figure 1.

The accuracy of the factorization is expected to be $\sim 1/N_c^2$, $N_c = 3$ is the color number. (The $1/N_c^2$ ambiguity of the $D = 8$ quark-gluon condensate factorization was explicitly demonstrated in [24].) Another version of the factorization, which employs the heavy quark expansion, was proposed in [25].

The ratio r_n with the operators (4.7) is shown by the hatched curves in the figure 2 for the $D = 4$ gluon condensate $\langle G^2\rangle/(4\bar{m}^2)^2 = 2\times10^{-4}$. Comparing the figures 1 and 2 one sees, that the contribution of the $\langle G^4\rangle$ operators in the factorization model is smaller than in the instanton one. It allows to establish certain stability region, where the ratio r_n remains almost unchanged. This region is clearly visible for $Q^2 = 8\bar{m}^2$: at $n = 20 - 26$ the ratio is $r_n = 3.40 \pm 0.01$, which corresponds to the c-quark mass $\bar{m} = 1.260 \pm 0.005\,\mathrm{GeV}$. This mass is computed for the condensate $\langle aG^2\rangle = 0.008\,\mathrm{GeV}^4$.

In the same way the mass can be computed for other values of the condensate. A restriction on the charm quark mass for different condensates is calculated in the next section.

5. Restrictions on the c-quark mass and $D = 4$ gluon condensate

As the main result of the pseudoscalar charmonium sum rule (1.5), we may establish certain restrictions for the c-quark mass $\bar{m}$ for a given condensate $\langle aG^2\rangle$.

At first, let us neglect the higher dimension operators $\langle G^3\rangle$ and $\langle G^4\rangle$. The calculation goes as follows. For a given $Q^2/(4\bar{m}^2)$ one should establish the range of n, where the perturbation theory as well as operator expansion can be trusted. It is reasonable to require, that the perturbative corrections may not exceed $30 - 40\%$ of the leading term. The most dangerous is the α_s-correction to the gluon condensate contribution $\bar{M}_n^{(G,1)}$. Keeping in mind typical size of the QCD couping $\alpha_s/\pi \sim 0.1$, let us impose the restriction

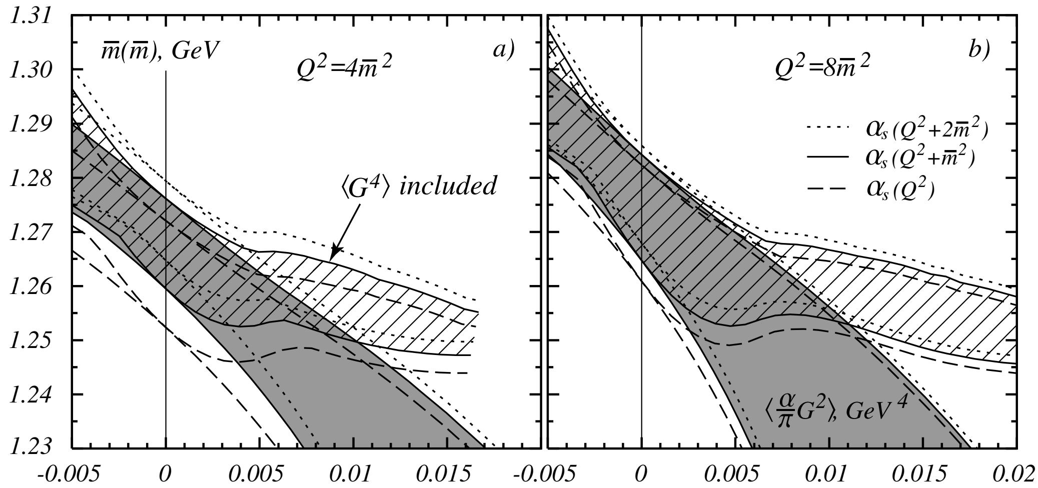

Figure 3: Charm quark mass $\bar{m}(\bar{m})$ versus $\langle aG^2\rangle$ obtained from the pseudoscalar sum rule. Shaded area displays the acceptable region when the $\langle G^4\rangle$ condensates are neglected. Hatched region corresponds to the $\langle G^4\rangle$ condensates, computed by factorization model. Dashed lines show the region boundaries for 2 alternative choices of the α_s scale.

$|\bar{M}_n^{(G,1)}/\bar{M}_n^{(G,0)}| < 4$. From the table 1 we find the following range of n:

$$n = 14 - 19 \qquad \text{for} \qquad \frac{Q^2}{4\bar{m}^2} = 1 \qquad \text{and} \qquad n = 22 - 30 \qquad \text{for} \qquad \frac{Q^2}{4\bar{m}^2} = 2\,. \tag{5.1}$$

The perturbative corrections to the moments are also tolerable in this region: the first correction $|\bar{M}_n^{(1)}/\bar{M}_n^{(0)}| < 2.5$ and the NNLO correction $|\bar{M}_n^{(2)}/\bar{M}_n^{(0)}| < 17$. Then, we take some value of $\langle aG^2\rangle/(4\bar{m})^2$ and find the maximal and minimal value of the ratio $r_n(Q^2)$ within this range of n. From these numbers we find the minimal and maximal values of the charm quark mass $\bar{m}$.

The results are shown by the shaded regions in the figure 3. Figure 3(a) and 3(b) display the restrictions, obtained from the sum rule (4.1) at $Q^2 = 4\bar{m}^2$ and $Q^2 = 8\bar{m}^2$ respectively. Since unknown higher order in α_s moments are discarded everywhere, the results depend on the choice of the scale, at which α_s is taken. The dark area shows the acceptable region for the scale (4.3). The dashed and dotted lines display the boundaries of the acceptable region, if $\bar{m}^2$ is added to or subtracted from this scale. The scale dependence is weaker at $Q^2 = 8\bar{m}^2$.

It is clear from the figure 3, that the pseudoscalar sum rule prefers lower values of the gluon condensate. In particular, for the mass $\bar{m} = 1.275 \pm 0.015\,\text{GeV}$ [3] one obtains the upper condensate limit $\langle aG^2\rangle < 0.008\,\text{GeV}^4$.

Now let us include the higher dimension operators. As follows from the instanton model analisys, the contribution of the $D = 6$ condensate $\left\langle G^3\right\rangle$ is small in the region of interest. But the $D = 8$ operators change the ratio r_n essentially. At some n their contribution exceeds the leading condensate $\left\langle G^2\right\rangle$; at this point the OPE series is divergent. Let us require that the contribution of the $D = 8$ operators $O_4^{(i)}$ to the moments may not exceed 30% of the $D = 4$ condensate contibution. This requirement further reduces the region of

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n (5.1) depending on the condensate size. In the factorization model $\langle G^4\rangle \sim (\langle G^2\rangle)^2$, and the region becomes smaller for higher condensate $\langle G^2\rangle$.

The hatched regions in the figure 3 display the inclusion of the $\langle G^4\rangle$ operators in the factorization model. For $\langle aG^2\rangle > 0.005\,\mathrm{GeV}^4$ the $D = 8$ operators change the ratio r_n drastically. They compensate the leading condensate and the ratio r_n becomes almost independent of the $\langle aG^2\rangle$ condensate in the stability region. From the hatched area in the figure 3 one gets the following limits of the c-quark mass:

$$\bar{m}_c(\bar{m}_c) = 1.26 \pm 0.02\,\mathrm{GeV} \tag{5.2}$$

indenpendently on the condensate value. The mass (5.2) is in agreement with the result $\bar{m} = 1.275 \pm 0.015\,\mathrm{GeV}$, obtained from the vector charmonium sum rules in [3].

If the $\langle G^4\rangle$ operators are included, it becomes rather difficult to obtain certain restrictions on the condensate value. However, for large condensate the stability region is narrow, and the results become unreliable. In particular, for $\langle aG^2\rangle > 0.015\,\mathrm{GeV}^4$ and $Q^2 = 4\bar{m}^2$ there is no region of n, where the OPE series looks convergent. This sets the natural limit of the condensate value, at which the pseudoscalar sum rule works.

Acknowledgments

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A. Approximations for $R^{(2),p}$

Let us define the dimensionless coefficient functions for the perturbative correlator (2.1) as follows:

$$\Pi^p(q^2) = 4m^2 q^2 \sum_{k\geq 0} \Pi^{(k),p}(q^2) a(m^2)\,, \qquad \Pi^{(k),p}(q^2) = \frac{3q^2}{8\pi^2}\int_0^\infty \frac{R^{(k),p}(s,m^2)}{s(s-q^2)} ds\,, \tag{A.1}$$

for definiteness we take the QCD coupling at the scale $\mu^2 = m^2$ and put the constants $c_0 = c_1 = 0$ in the dispersion relation (1.3). The 3-loop function $\Pi^{(2),p}$ is decomposed into 5 gauge invariant parts in the same way as $R^{(2),p}$ (2.4).

At first we consider the nonabelian part $\Pi^{(2),p}_{NA}$. Its expansion near $z \equiv q^2/(4m^2) = 0$ until z^8 is available in [10]. Then, as usual, we reexpand this series in terms of the variable ω, which naturally appears in the perturbative calculations:

$$\omega = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\,. \tag{A.2}$$

The expansion of the polarization operator in ω has appropriate analytical properties, namely the cut at $z = [1, \infty)$. In many cases the Pade approximation was proved to have better accuracy, than Tailor series. The best results (see the discussion in Section 2) were obtained for the Pade approximation [5/2]:

$$\Pi^{(2),p}_{NA}(\omega) = \frac{3}{16\pi^2}\times \\ \times\frac{-7.43220\omega+73.5001\omega^2+2.69248\omega^3-13.3868\omega^4-0.91032\omega^5-1.61120\omega^6}{1-0.18579\omega-0.63849\omega^2}\,. \tag{A.3}$$

The accuracy of the Pade approximated abelian part $\Pi^{(2),p}_{A}$ is worse because of Coulomb behavior $\sim 1/\sqrt{1-z}$ near the threshold. It turns out, however, that the expansion in ω converges faster, if the multiplier $1/(1-\omega^2)$ is separated out:

$$\begin{aligned}\Pi^{(2),p}_{A}(\omega) &= \frac{3}{16\pi^2}\Big[10.8487\,\omega+90.9348\,\omega^2+111.510\,\omega^3+73.3467\,\omega^4+ \\ &\qquad +113.363\,\omega^5+74.8959\,\omega^6+114.366\,\omega^7+77.2105\,\omega^8+O(\omega^9)\Big] \\ &= \frac{3}{16\pi^2}\frac{1}{1-\omega^2}\Big[10.8487\,\omega+90.9348\,\omega^2+100.662\,\omega^3-17.5881\,\omega^4+ \\ &\qquad +1.85289\,\omega^5+1.54919\,\omega^6+1.00228\,\omega^7+2.31461\,\omega^8+O(\omega^9)\Big]\,.\end{aligned}$$

Now we construct the Pade approximation, which well reproduces all asymptotic and first 8 moments from the dispersion relation:

$$\Pi^{(2),p}_{A}(\omega) = \frac{3}{16\pi^2}\times \\ \times\frac{10.8487\,\omega+145.611\,\omega^2+507.376\,\omega^3+57.361\,\omega^4-565.406\,\omega^5+94.514\,\omega^6}{(1-\omega^2)\,(1+5.03984\,\omega-4.75471\,\omega^2)}\,. \tag{A.4}$$

Few more work should be done to construct the approximation for the singlet polarization function $\Pi^{(2),p}_{S}$. Its expansion near $z=0$ until z^8 is available in [11]. The singlet correlator contains intermediate massless 2-gluon state, so the cut starts from $z=0$, the expansion in [11] has the terms $\sim \ln(-z)$ and the conformal mapping procedure ($z\to\omega$) is not applicable here. As discussed in section 2, in our sum rules we use the polarization operator *without 2-gluon cut* (3.2), so the correspondent part of the polarization operator should be subtracted from the result of [11]:

$$\Pi^{(2),p}_{gg} = \frac{3}{16\pi^2}C_F T(-y)\int_0^\infty \frac{|f(z)|^2}{z^2(z+y)}dz\,, \qquad y=\frac{Q^2}{4m^2}\,, \tag{A.5}$$

where the function $f(z)$ is given in (2.7). The integral from $z=1$ to ∞ is regular at $y=0$ and can be expanded by y in Tailor series. But the integral from $z=0$ to 1 requires special care, since it behaves as $\sim \ln y$ at $y\to 0$. In order to obtain the expansion for small y, we suggest to use the following series for the function $|f(z)|^2$ for $0<z<1$ [26]:

$$\arcsin^4\!\sqrt{z} = \frac{3}{2}\sum_{n=2}^{\infty} z^n\,\frac{(n-1)!}{n\,(1/2)_n}\sum_{k=1}^{n-1}\frac{1}{k^2}\,. \tag{A.6}$$

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Then we obtain the following expansion:

$$\int_0^1 \frac{\arcsin^4\!\sqrt{z}}{z^2(z+y)}dz \;=\; -\frac{\operatorname{arcsh}^4\!\sqrt{y}}{y^2}\,\ln y + \sum_{n=0}^{\infty}(-y)^n\, I_n\,, \tag{A.7}$$

where the constants

$$I_n = \frac{3}{2}\sum_{k=2,\neq n+2}^{\infty}\frac{(k-1)!}{k(k-n-2)\,(1/2)_k}\sum_{j=1}^{k-1}\frac{1}{j^2}$$

can be computed analytically for any n with the help of recursive relations:

$$\begin{aligned} I_n \;=\; & -\frac{\pi^4}{16(n+2)} + \frac{2\,(n+1)!}{(n+2)\,(5/2)_n}\times \\ & \times\left\{\pi^2\ln 2 - \frac{7}{2}\zeta_3 + \sum_{k=1}^{n+1}\left[\frac{1}{k^2(n+2)} + \frac{2}{k^3} - \frac{(1/2)_k}{k^2\,(k-1)!}\frac{\pi^2-8}{2} + \right.\right. \\ & \left.\left. \qquad + \frac{1}{k^2}\sum_{j=1}^{n+1}\left(\frac{1}{j+1/2}-\frac{1}{j}\right) - \frac{1}{k^2}\sum_{j=1}^{k-1}\frac{(3/2)_j}{j^2\,(j-1)!}\right]\right\}. \end{aligned} \tag{A.8}$$

Now one obtains regular at $y = 0$ Tailor expansion of the polarization operator $\Pi_S^{(2),p}$ without the 2-gluon cut, applies the conformal mapping and constructs the Pade approximation:

$$\begin{aligned} \Pi_{S\text{-}gg}^{(2),p}(\omega) &\equiv \Pi_S^{(2),p} - \frac{1}{C_F T}\Pi_{gg}^{(2),p} \\ &= \frac{3}{16\pi^2}\,\frac{-7.86155\omega+6.98952\omega^2+3.24217\omega^3-1.68013\omega^4-0.31547\omega^5-0.00464\omega^6}{1-0.05236\omega-0.32689\omega^2}. \end{aligned} \tag{A.9}$$

Eventually we take the imaginary part of (A.3), (A.4), (A.9) and obtain the correspondent coefficient functions $R^{(2),p}$:

$$\begin{aligned} R_{NA}^{(2),p} &= \frac{8\pi}{3}\,\mathrm{Im}\,\Pi_{NA}^{(2),p}(\omega) + \frac{11}{16}\,R^{(1),p}\ln\frac{\mu^2}{m^2}\,, \\ R_A^{(2),p} &= \frac{8\pi}{3}\,\mathrm{Im}\,\Pi_A^{(2),p}(\omega)\,, \\ R_S^{(2),p} - \frac{1}{C_F T}R_{gg}^{(2),p} &= \frac{8\pi}{3}\,\mathrm{Im}\,\Pi_{S-gg}^{(2),p}(\omega)\,, \qquad \text{where} \qquad \omega = \frac{1+i\sqrt{z-1}}{1-i\sqrt{z-1}}. \end{aligned} \tag{A.10}$$

B. α_s-correction to the condensate contribution

The α_s correction to the $D = 4$ gluon condensate contribution was found in [16]. Let us parametrize it by dimensionless function $f^{(1),p}(z)$:

$$\Pi_{\mathrm{OPE}}^p(s) = \cdots + \frac{O_2}{4\pi^2}\,a\,f^{(1),p}(z)\,, \tag{B.1}$$

where dots denote the leading order operator contribution (2.8). Here we construct a dispersion-like relation for this function, convenient for numerical calculation of the moments. We will follow the method, used in [3] for the vector current correlator.

The imaginary part is:

$$
\mathrm{Im} f^{(1),p}(z+i0) = \frac{\pi}{96 z^3 v^5}\left[P_2^P(z) + \frac{P_3^P(z)}{zv}\ln\frac{1-v}{1+v} + P_4^P(z)(1-z)\left(2\ln v + \frac{3}{2}\ln(4z)\right)\right], \tag{B.2}
$$

where the polynomials $P_i^P(z)$ are given in [16, table 1]. Taking the contour integral in z-plane around the cut $z=[1,\infty)$, one could write down the following dispersion relation:

$$
\begin{aligned}
f^{(1),p}(t) &= \frac{1}{\pi}\int_{1+\epsilon}^{\infty}\frac{\mathrm{Im}\, f^{(1),p}(z+i0)}{z-t}dz + \sum_{i=1}^{3}\frac{\pi^2 f_i^p}{(1-t)^i} - \frac{11}{384}\frac{\epsilon^{-3/2}}{1-t} + \\
&+ \left[\frac{15473}{6912} + \frac{35}{36}\ln(8\epsilon)\right]\frac{\epsilon^{-1/2}}{1-t} + \frac{11}{128}\frac{\epsilon^{-1/2}}{(1-t)^2},
\end{aligned} \tag{B.3}
$$

where $\epsilon \to 0$ and

$$
f_1^p = -\frac{11}{256}, \qquad f_2^p = \frac{69}{256}, \qquad f_3^p = -\frac{197}{2304}. \tag{B.4}
$$

To simplify the calculation, we represent the imaginary part (B.2) in the following form:

$$
\frac{1}{\pi}\mathrm{Im}\, f^{(1),p}(z+i0) = F_1^p(z) + F_2^{p\prime}(z) + \frac{1}{2}F_3^{p\prime\prime}(z), \tag{B.5}
$$

where the functions $F_p^i(z)$ grow not faster than $(z-1)^{-1/2}$ at $z\to 1$ and have appropriate asymptotic at infinity. Our choice is:

$$
\begin{aligned}
F_1^p(z) &= \frac{1}{96\, z^2 v}\left[56 + \frac{49}{3}z + 552z^2 + \left(\frac{207}{4} - 60z + 904z^2 - 4216z^3 + 3312z^4\right)\times\right.\\
&\left.\times\frac{1}{zv}\ln\frac{1-v}{1+v} + \left(\frac{140}{3} + \frac{76}{3}z - \frac{5120}{3}z^2 + 2208z^3\right)\left(2\ln v + \frac{3}{2}\ln(4z)\right)\right]\\
F_2^p(z) &= \frac{1}{96\, z^2 v}\left[\frac{11}{4} + \frac{645}{2}z + \left(-\frac{197}{24} + \frac{1439}{24}z\right)\frac{1}{zv}\ln\frac{1-v}{1+v} + \right.\\
&\left. + \frac{280}{3}z\left(2\ln v + \frac{3}{2}\ln(4z)\right)\right]\\
F_3^p(z) &= \frac{1}{96\, zv}\left[-\frac{131}{6} - \frac{197}{12\, zv}\ln\frac{1-v}{1+v}\right].
\end{aligned} \tag{B.6}
$$

Then, the r.h.s. of (B.3) can be integrated by parts twice, all divergent in $\epsilon \to 0$ terms cancel and the dispersion relation can be brought to the following form:

$$
f^{(1),p}(t) = \sum_{i=1}^{3}\left[\frac{\pi^2 f_i^p}{(1-t)^i} + \int_1^{\infty}\frac{F_i^p(z)}{(z-t)^i}dz\right]. \tag{B.7}
$$

C. α_s-corrections to the moments

Numerical values of the perturbative corrections to the moments $\bar{M}_n^{(k),p}/\bar{M}_n^{(0),p}$ and the α_s-correction to the condensate contribution $\bar{M}_n^{(O2)(1),p}/\bar{M}_n^{(O2),p}$ are given in the table 1 for

Q^2	$\bar{M}_n^{(1),p}/\bar{M}_n^{(0),p}$			$\bar{M}_n^{(2),p}/\bar{M}_n^{(0),p}$			$\bar{M}_n^{(O2)(1),p}/\bar{M}_n^{(O2),p}$		
n	0	$4\bar{m}^2$	$8\bar{m}^2$	0	$4\bar{m}^2$	$8\bar{m}^2$	0	$4\bar{m}^2$	$8\bar{m}^2$
2	2.357	0.825	−0.176	−1.132	−11.89	−16.24	1.489	3.506	3.617
3	3.87	2.976	2.064	12.85	−0.466	−8.633	−1.58	3.161	3.731
4	3.976	3.894	3.179	19.28	9.077	−0.018	∞	2.486	3.611
5	3.508	4.309	3.831	20.86	15.63	7.025	3.91	1.47	3.32
6	2.716	4.442	4.226	19.8	19.83	12.51	0.34	−0.021	2.88
7	1.71	4.391	4.454	17.56	22.25	16.68	−1.955	−2.424	2.291
8	0.55	4.209	4.562	15.11	23.33	19.75	−3.979	−7.447	1.536
9	−0.727	3.927	4.581	13.18	23.42	21.93	−5.923	−31.87	0.567
10	−2.098	3.568	4.529	12.33	22.76	23.37	−7.847	31.4	−0.713
11	−3.545	3.146	4.42	13.	21.58	24.19	−9.773	11.7	−2.508
12	−5.057	2.671	4.264	15.54	20.04	24.51	−11.71	6.907	−5.312
13	−6.623	2.152	4.068	20.24	18.29	24.41	−13.66	4.376	−10.66
14	−8.237	1.594	3.837	27.37	16.44	23.97	−15.63	2.6	−26.79
15	−9.892	1.003	3.577	37.13	14.6	23.26	−17.62	1.16	338.4
16	−11.58	0.383	3.29	49.72	12.85	22.33	−19.62	−0.105	27.93
17	−13.31	−0.264	2.98	65.29	11.27	21.23	−21.64	−1.269	15.2
18	−15.07	−0.934	2.649	84.01	9.935	20.01	−23.67	−2.373	10.4
19	−16.85	−1.626	2.3	106.	8.893	18.7	−25.72	−3.439	7.707
20	−18.66	−2.336	1.933	131.4	8.205	17.34	−27.78	−4.479	5.863
21	−20.49	−3.064	1.551	160.3	7.917	15.97	−29.85	−5.504	4.447
22	−22.34	−3.809	1.155	192.8	8.075	14.6	−31.93	−6.519	3.274
23	−24.21	−4.568	0.745	229.	8.718	13.27	−34.03	−7.527	2.252
24	−26.1	−5.341	0.324	268.9	9.884	12.	−36.14	−8.531	1.327
25	−28.01	−6.128	−0.109	312.8	11.61	10.81	−38.26	−9.533	0.47
26	−29.93	−6.926	−0.552	360.5	13.91	9.728	−40.38	−10.53	−0.339
27	−31.87	−7.735	−1.005	412.3	16.84	8.76	−42.52	−11.54	−1.114
28	−33.82	−8.555	−1.467	468.1	20.41	7.929	−44.66	−12.54	−1.864
29	−35.78	−9.385	−1.938	528.1	24.65	7.251	−46.82	−13.55	−2.595
30	−37.76	−10.22	−2.417	592.2	29.58	6.74	−48.98	−14.55	−3.311

Table 1: α_s-corrections to PS-moments in $\overline{\mathrm{MS}}$ scheme.

$Q^2/(4\bar{m}^2) = 0, 1, 2$ and $n = 2 - 30$. The coefficient functions $\bar{M}_n^p$ are defined in $\overline{\mathrm{MS}}$ scheme according to (3.8) and (3.9). Remind, that the expansion (3.8) goes by $a(\bar{m}^2)$. If one takes the QCD coupling at another scale, the function $\bar{M}_n^{(2),p}$ must be changed according to (3.10).

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